Belt-Driven Load and Motor: Modeling, Analysis, and Control

Physical System

Motor

Load

Torque \( T_M \) \( \dot{\theta}_M \)

\( J_M \) inertia of motor + pulley

\( B_M \) viscous damping - motor side

\( J_L \) inertia of load + pulley

\( B_L \) viscous damping - load side

\( N = \frac{R_L}{R_M} \)
Case I: Rigid Belt (No Compliance)

Free-Body Diagrams

\[ R_M \theta_M = R_L \theta_L \]
\[ N = \frac{R_L}{R_M} \]
\[ \theta_M = N \theta_L \]

One-Degree-of-Freedom System

Equation of Motion in terms of either \( \theta_M \) or \( \theta_L \)

\[ J_{eq,m} \ddot{\theta}_m + B_{eq,m} \dot{\theta}_m = T_{eq,m} \]
\[ J_{eq,l} \ddot{\theta}_l + B_{eq,l} \dot{\theta}_l = T_{eq,l} \]

\[ T_{eq,m} = T_m(N) \]
\[ J_{eq,m} = J_m(N^2) \]
\[ B_{eq,m} = B_m(N^2) \]

\[ T_{eq,l} = T_l + J_m(N^2) \]
\[ B_{eq,l} = B_l + B_m(N^2) \]
Note that this situation is the same as an ideal gear train where \[ N = \frac{\omega_m}{\omega_L} \]

\[
\begin{align*}
T_m & \quad \Theta_m \\
\theta_m & \quad \beta_m \\
\hline
N & \quad \omega_L \quad \beta_L
\end{align*}
\]

One-Degree-of-Freedom System

Equation of Motion

\[
\left( J_L + N^2 J_m \right) \ddot{\omega}_L + \left( \beta_L + N^2 \beta_m \right) \dot{\omega}_L = N T_m
\]

or

\[
\left( J_m + \frac{1}{N^2} J_L \right) \ddot{\omega}_m + \left( \beta_m + \frac{1}{N^2} \beta_L \right) \dot{\omega}_m = T_m
\]

So, for an ideal gear train \[ N = \frac{\omega_m}{\omega_L} \] (opposite directions)

- for an ideal belt-driven system \[ N = \frac{\omega_m}{\omega_L} \] (same direction)
Consider Control: PI controller

\[
\frac{\dot{\theta}}{\theta_c} = \frac{G_c G}{1 + G_c G H}
\]

Characteristic Equation: \(1 + G_c G H = 0\)

\[
1 + \left( \frac{K_p s + K_i}{s} \right) \left( \frac{1}{J_{eg} s + B_{eg}} \right) = 0
\]

\[
(J_{eg} s^2 + B_{eg} s) + K_p s + K_i = 0
\]

\[
s^2 + \left( \frac{B_{eg} + K_p}{J_{eg}} \right) s + \left( \frac{K_i}{J_{eg}} \right) = 0
\]
\[ s^2 + \alpha, s + \alpha_0 = 0 \]

**Motor Side**

\[ \alpha_1 = \frac{B_m R_e^2}{J_{eq,M}} + \frac{K_p R_e^2}{J_{eq,M}} = \frac{B_m R_e^2 + B_L R_m^2 + K_p R_e^2}{J_m R_e^2 + J_L R_m^2} \]

\[ \alpha_0 = \frac{K_i}{J_{eq,M}} = \frac{K_i R_e^2}{J_m R_e^2 + J_L R_m^2} \]

**Load Side**

\[ s^2 + \beta, s + \beta_0 = 0 \]

**Characteristic Equation:**

\[ 1 + \left( \frac{K_p s + K_i}{s} \right) (N) \left( \frac{1}{J_{eq,L} s + B_{eq,L}} \right) = 0 \]

\[ (J_{eq,L} s^2 + B_{eq,L} s) + N (K_p s + K_i) = 0 \]

\[ s^2 + \left( \frac{B_{eq,L} + NK_p}{J_{eq,L}} \right) s + \left( \frac{NK_i}{J_{eq,L}} \right) = 0 \]
\[ s^2 + \beta_1 s + \beta_0 = 0 \]

\[ \beta_1 = \frac{B_{eg,c} + N K_p}{T_{eg,c}} = \frac{B_L R_m^2 + B_M R_e^2 + K_p R_m}{J_L R_m^2 + J_M R_e^2} \]

\[ \beta_0 = \frac{N K_i}{T_{eg,c}} = \frac{K_i R_e R_m}{J_L R_m^2 + J_M R_e^2} \]

In either case, the gains of the PI controller may be chosen to place the poles of the characteristic equations appropriately, which is preferable?
Case II: Compliant Belt

Assumption: Transmission of power takes place on the tight side. The net change in tension on the slack side is negligible. Model the tight side of the belt as a spring/damper (K and B).

Free-Body Diagrams

Motor

\[ T_M \]

\[ J_H \ddot{\omega}_H \]

\[ (R_M \dot{\omega}_M - R_L \dot{\omega}_L) K R_H \]

\[ B_H \dot{\omega}_H \]

Load

\[ \Theta_L \]

\[ J_L \ddot{\omega}_L \]

\[ (R_M \dot{\omega}_M - R_L \dot{\omega}_L) K R_L \]

\[ (R_M \dot{\omega}_M - R_L \dot{\omega}_L) B R_L \]
Equations of Motion

\[ T_M = J_M \ddot{\theta}_M + B_M \dot{\theta}_M + K R_M (R_M \dot{\theta}_M - R_L \dot{\theta}_L) + B R_M (R_M \ddot{\theta}_M - R_L \ddot{\theta}_L) \]

\[ K R_L (R_M \dot{\theta}_M - R_L \dot{\theta}_L) + B R_L (R_M \ddot{\theta}_M - R_L \ddot{\theta}_L) = \tau_L \ddot{\theta}_L + B \dot{\theta}_L \dot{\theta}_L \]

Note that the potential energy stored in the belt is:

\[ V(\theta_M, \theta_L) = \frac{1}{2} K [R_M \theta_M - R_L \theta_L]^2 \]

Transport of belt material is taking place on the slack side and transmission of power is taking place on the tight side.
Laplace Transform: Transfer Functions and Block Diagrams

\[ T_M(s) = \left[ J_M s^2 + (B_m + BR_m^2)s + (KR_m^2) \right] \Theta_M(s) + \left[ -BR_m R_L s - KR_m R_L \right] \Theta_L(s) \]

\[ \Theta_L(s) = \left[ J_L s^2 + (B_L + BR_L^2)s + (KR_L^2) \right] \Theta_M(s) + \left[ -BR_m R_L s - KR_m R_L \right] \Theta_L(s) \]

\[
\begin{bmatrix}
J_M s^2 + (B_m + BR_m^2)s + KR_m^2 & -BR_m R_L s - KR_m R_L \\
-BR_m R_L s - KR_m R_L & J_L s^2 + (B_L + BR_L^2)s + KR_L^2
\end{bmatrix}
\begin{bmatrix}
\Theta_M(s) \\
\Theta_L(s)
\end{bmatrix} =
\begin{bmatrix}
T_M(s) \\
0
\end{bmatrix}
\]

\[
\frac{\Theta_M(s)}{T_M(s)} = \frac{J_L s^2 + (B_L + BR_L^2)s + KR_L^2}{D(s)}
\]

\[
\frac{\Theta_L(s)}{T_M(s)} = \frac{BR_m R_L s + KR_m R_L}{D(s)}
\]

\[ D(s) = (J_M J_L)s^4 + \left[ B (J_L R_m^2 + J_M R_L^2) + J_M B_L + J_L B_m \right] s^3 + \left[ K (J_L R_m^2 + J_M R_L^2) + B_m B_L + B (B_L R_m^2 + B_m R_L^2) \right] s^2 + \left[ K (B_L R_m^2 + B_m R_L^2) \right] s \]
Verification:

\[
\begin{align*}
\left[ \tau_m - (\omega_m - N\omega_L)(BR_m^2 + \frac{KR_m^2}{s}) \right] \left[ \frac{1}{J_M s + B_M} \right] &= \omega_M \\
(J_M s + B_M) \omega_M &= \tau_m - \omega_M BR_m^2 - \omega_M KR_m^2 + \left( \frac{R_L}{R_m} \right) \omega_L BR_m^2 + \left( \frac{R_L}{R_m} \right) \omega_L KR_m^2 \\
J_M \ddot{\omega}_M + B_M \dot{\omega}_M + KR_m (R_m \omega_M - R_L \omega_L) + BR_m (R_m \ddot{\omega}_M - R_L \ddot{\omega}_L) &= \tau_m \quad \checkmark \\
\left\{ (\omega_M - N\omega_L) \left[ BR_m^2 + \frac{KR_m^2}{s} \right] N - \tau_L \right\} \left( \frac{1}{J_L s + B_L} \right) &= \omega_L \\
(J_L s + B_L) \omega_L &= -\tau_L + N \left[ BR_m^2 \omega_M - N BR_m^2 \omega_L + KR_m^2 \omega_M - N KR_m^2 \omega_L \right] \\
J_L \ddot{\omega}_L + B_L \dot{\omega}_L &= -\tau_L + KR_L (R_m \omega_M - R_L \omega_L) + BR_L (R_m \ddot{\omega}_M - R_L \ddot{\omega}_L) \quad \checkmark
\end{align*}
\]
To simplify the mathematics and since $b_L$ and $b_M$ have little effect on the resonance/anti-resonance behavior of the system, let $b_L = 0$ and $b_M = 0$. Also note that Coulomb friction has been neglected. The fixed value of Coulomb friction has little impact on stability when the motor is moving. At rest, the impact of stiction is thought of as increasing the load inertia when the motor is at rest. This accounts for the tendency of systems to change resonance behavior when the motion stops.
with $b_m = 0$ and $b_L = 0$, the transfer functions become:

\[
\begin{align*}
\Theta_M(s) &= \frac{1}{(J_L R_m^2 + J_m R_L^2) s^2} \left[ \frac{J_L s^2 + BR_L s + K R_L^2}{\frac{J_m J_L}{J_L R_m^2 + J_m R_L^2} s^2 + B s + K} \right] \\
\Theta_L(s) &= \frac{1}{(J_L R_m^2 + J_m R_L^2) s^2} \left[ \frac{B R_m R_L s + K R_m R_L}{\frac{J_m J_L}{J_L R_m^2 + J_m R_L^2} s^2 + B s + K} \right]
\end{align*}
\]
These transfer functions can be written in standard form. (Here $b_m = 0$ and $b_l = 0$)

\[
\frac{\Theta_m(s)}{T_m(s)} = \frac{K_1 \left[ \frac{s^2}{\omega_A^2} + \frac{2S_A s}{\omega_A} + 1 \right]}{s^2 \left[ \frac{s^2}{\omega_K^2} + \frac{2S_K s}{\omega_K} + 1 \right]}
\]

\[
\frac{\Theta_l(s)}{T_m(s)} = \frac{K_2 (TS + 1)}{s^2 \left[ \frac{s^2}{\omega_K^2} + \frac{2S_K s}{\omega_K} + 1 \right]}
\]

\[
K_2 = \frac{R_m R_l}{J_m R_l^2 + J_l R_m^2}
\]

\[
K_1 = \frac{R_l^2}{J_m R_l^2 + J_l R_m^2}
\]

\[
\tau = \frac{B}{K}
\]

\[
\omega_A = \left[ \frac{K R_l^2}{J_l} \right]^{\frac{1}{2}}
\]

\[
S_A = \frac{B}{2} \left[ \frac{R_l}{K J_l} \right]^{\frac{1}{2}}
\]

\[
\omega_K = \left[ \frac{K (J_m R_l^2 + J_l R_m^2)}{J_m J_l} \right]^{\frac{1}{2}}
\]

\[
S_K = \left[ \frac{K J_m J_l}{J_m R_l^2 + J_l R_m^2} \right]^{\frac{1}{2}}
\]
Observations (proofs not shown)

1) When $w_L$ is used for feedback, it can be shown that the closed-loop system characteristic equation is 4th order and the coefficients of the $s^3$ and $s^2$ terms do not depend on the PI controller gains, $K_p$ and $K_i$. Thus it may not be possible to place the poles of the characteristic equation at desired locations. However, when $w_m$ is used for feedback, the closed-loop system characteristic equation is again 4th order but the coefficients of the $s^3$ and $s^2$ terms now do depend on the PI controller gains, $K_p$ and $K_i$.

2) The closed-loop system with $w_m$ feedback is stable and $w_m \rightarrow (w_m)_c$ for all $K_p > 0$ and $K_i > 0$. In addition $w_L \rightarrow \left(\frac{R_m}{R_L}\right)(w_m)_c$. 
3) For \( w_L \) feedback, the same conclusion as in (2) can not be stated.

4) Consider the equation of motion for the load:
\[
J_L \ddot{\theta}_L + (B_L + BR_L^2) \dot{\theta}_L + KR_L^2 \theta_L = BR_L R_M \dot{\theta}_M + KR_L R_M \theta_M
\]
Differentiate this equation:
\[
J_L \dddot{\theta}_L + (B_L + BR_L^2) \ddot{\theta}_L + KR_L^2 \ddot{\theta}_L = BR_L R_M \dddot{\theta}_M + KR_L R_M \ddot{\theta}_M
\]
This equation shows that \( w_L \) can attain steady state only when \( w_M \) attains steady state first. Even after \( w_M \) attains steady state, \( w_L \) continues to exhibit damped oscillations for some time before it attains steady state.
Thus, by measuring only $w_2$ and using a PI controller, we will not be able to say for sure whether the oscillations in $w_2$ are due to fluctuations in $w_1$, or the oscillations are indeed damped oscillations. In such a situation, the controller attempts to react to the damped oscillations also, and in this process, changes $w_1$, which in turn affects $w_2$ because of the dynamics given by the equation. This process of correcting $w_2$ may go on for a very long time, if not forever, depending on the damping present on the load side. Thus measuring $w_2$ for control does not present a desirable situation.
Let's compare this to the direct-drive with a flexible coupling.

Equations of Motion:

\[ \tau_m = J_m \ddot{\theta}_m + B_m \dot{\theta}_m + K (\theta_m - \theta_L) + B (\ddot{\theta}_m - \ddot{\theta}_L) \]

\[ \tau_L = J_L \ddot{\theta}_L + B_L \dot{\theta}_L - K (\theta_m - \theta_L) - B (\ddot{\theta}_m - \ddot{\theta}_L) \]

Compare to Equations of Motion of Belt-Driven System (pg. 7):

\[ \tau_m = J_m \ddot{\theta}_m + B_m \dot{\theta}_m + K_R (R_m \dot{\theta}_m - R_L \dot{\theta}_L) + B_R (R_m \ddot{\theta}_m - R_L \ddot{\theta}_L) \]

\[ \tau_L = J_L \ddot{\theta}_L + B_L \dot{\theta}_L - K_R (R_m \dot{\theta}_m - R_L \dot{\theta}_L) - B_R (R_m \ddot{\theta}_m - R_L \ddot{\theta}_L) \]
We can write these as (using $N = \frac{R_l}{R_m}$):

\[
\tau_m = J_m \ddot{\theta}_m + b_m \dot{\theta}_m + KR_m^2 (\dot{\theta}_m - N \dot{\theta}_l) + BR_m^2 (\ddot{\theta}_m - N \ddot{\theta}_l)
\]

\[0 = J_l \ddot{\theta}_l + b_l \dot{\theta}_l - KR_l^2 \left( \frac{\dot{\theta}_m}{N} - \dot{\theta}_l \right) - BR_l^2 (\ddot{\theta}_m - \ddot{\theta}_l)\]

Let $N = 1$ ($R_l = R_m \equiv R$)

\[
\tau_m = J_m \ddot{\theta}_m + b_m \dot{\theta}_m + KR^2 (\dot{\theta}_m - \dot{\theta}_l) + BR^2 (\ddot{\theta}_m - \ddot{\theta}_l)
\]

\[0 = J_l \ddot{\theta}_l + b_l \dot{\theta}_l - KR^2 \left( \frac{\dot{\theta}_m}{N} - \dot{\theta}_l \right) - BR^2 (\ddot{\theta}_m - \ddot{\theta}_l)\]

A direct comparison shows:

\[J_m, J_l, b_m, b_l \rightarrow \text{same}\]

Coupling \(K \rightarrow KR^2\) \text{Belt-Driven}\n
\[b \rightarrow BR^2\]
Let's continue with the direct drive with the flexible coupling system.

Eqs of Motion:

\[ \ddot{T}_M = T_m \ddot{\theta}_M + b_m \dot{\theta}_M + K(\dot{\theta}_M - \dot{\theta}_L) + B(\ddot{\theta}_M - \ddot{\theta}_L) \]
\[ 0 = T_L \ddot{\theta}_L + b_L \dot{\theta}_L - K(\dot{\theta}_M - \dot{\theta}_L) - B(\ddot{\theta}_M - \ddot{\theta}_L) \]

Laplace Transform:

\[ T_m(s) = \left[ T_m s^2 + (b_m + b) s + K \right] \Theta_M(s) + \left[ -B s - K \right] \Theta_L(s) \]
\[ 0 = \left[ T_L s^2 + (b_L + b) s + K \right] \Theta_L(s) + \left[ -B s - K \right] \Theta_M(s) \]

\[
\begin{bmatrix}
T_m s^2 + (b_m + b) s + K \\
-(B s + K) \\
-(B s + K) \\
T_L s^2 + (b_L + b) s + K
\end{bmatrix}
\begin{bmatrix}
\Theta_M(s) \\
\Theta_L(s)
\end{bmatrix}
= 
\begin{bmatrix}
T_m(s) \\
0
\end{bmatrix}
\]

\[
\frac{\Theta_M(s)}{T_m(s)} = \frac{T_L s^2 + (b_L + b) s + K}{D(s)} \quad \frac{\Theta_L(s)}{T_m(s)} = \frac{B s + K}{D(s)}
\]

\[ D(s) = T_m T_L s^4 + [(T_m + T_L) b + T_m b_L + T_L b_M] s^3 + [(T_m + T_L) K + b_n b_L + B (b_L + b_M)] s^2 + [(b_n + b_L) K] s \]
Let $B_L = 0$ and $B_M = 0$ (as before):

\[
\frac{\Theta_M(s)}{T_M(s)} = \frac{1}{(T_M + J_L) s^2} \left[ \frac{J_L s^2 + B s + K}{J_L T_M} s^2 + B s + K \right] \]

\[
\frac{\Theta_L(s)}{T_M(s)} = \frac{1}{(T_M + J_L) s^2} \left[ \frac{B s + K}{J_L T_M} s^2 + B s + K \right] \]

\[
\frac{\Theta_M(s)}{T_M(s)} = K_1 \left[ \frac{s^2}{\omega_{AR}^2} + \frac{2 J_{AR} s}{\omega_{AR}} + 1 \right] \frac{s^2}{\omega_{AR}^2 + \frac{2 J_{AR} s}{\omega_{AR}} + 1} \]

\[
\frac{\Theta_L(s)}{T_M(s)} = \frac{K_2 (Ts + 1)}{s^2 \left[ \frac{s^2}{\omega_R^2} + \frac{2 J_{R} s}{\omega_R} + 1 \right]} \]

\[
K = \frac{1}{T_M + J_L} = K_2 \quad \omega_R = \left[ \frac{K (T_M + J_L)}{J_M J_L} \right]^{1/2} \quad \frac{1}{J_R} = \frac{B}{2} \left[ \frac{J_M J_L}{K T_M + J_L} \right]^{1/2} \]

\[
\omega_{AR} = \left[ \frac{K}{J_L} \right]^{1/2} \quad \frac{1}{J_{AR}} = \frac{B}{2} \left[ \frac{1}{K J_L} \right]^{1/2} \]
Block Diagram

\[
\begin{align*}
\sum \frac{1}{J_{M} s + \beta_{M}} \sum (s + \frac{K}{s}) \sum \frac{1}{J_{L} s + \beta_{L}} & \rightarrow \omega_{L} \\
\tau_{M} & \rightarrow \omega_{M}
\end{align*}
\]

Verification

\[
\left[ \tau_{M} - (\omega_{M} - \omega_{L})(s + \frac{K}{s}) \right] \left[ \frac{1}{J_{M} s + \beta_{M}} \right] = \omega_{M}
\]

\[
(J_{M} s + \beta_{M}) \omega_{M} = \tau_{M} - \omega_{M} \beta - \theta_{M} K + \omega_{L} \beta + K \theta_{L}
\]

\[
J_{M} \ddot{\omega}_{M} + \beta_{M} \dot{\omega}_{M} + K (\theta_{M} - \theta_{L}) + \beta (\dot{\omega}_{M} - \dot{\omega}_{L}) = \tau_{M} \quad \checkmark \quad \text{OK}
\]

\[
(\omega_{M} - \omega_{L})(s + \frac{K}{s}) \left( \frac{1}{J_{L} s + \beta_{L}} \right) = \omega_{L}
\]

\[
(J_{L} s + \beta_{L}) \omega_{L} = \beta \dot{\theta}_{M} - \beta \dot{\theta}_{L} + K \theta_{M} - K \theta_{L}
\]

\[
J_{L} \ddot{\theta}_{L} + \beta_{L} \dot{\theta}_{L} + \beta (\dot{\theta}_{L} - \dot{\theta}_{M}) + K (\theta_{L} - \theta_{M}) = 0 \quad \checkmark \quad \text{OK}
\]
One more situation to consider: gear train (ideal) + a flexible coupling.

\[ N = \frac{\Omega_M}{\Omega'_M} = \frac{w_M}{w'_M}, \]
\[ \theta_M' = N \theta_M, \]
\[ w_M' = \frac{w_M}{N} \]

Equations of Motion:

**Motor:**  
\[ J_M \ddot{\theta}_M + B_M \dot{\theta}_M + \frac{B}{N} (\frac{\ddot{\theta}_M}{N} - \ddot{\theta}_L) + \frac{K}{N} (\frac{\dot{\theta}_M}{N} - \theta_L) = T_M \]

**Load:**  
\[ J_L \ddot{\theta}_L + B_L \dot{\theta}_L - B (\frac{\dot{\theta}_M}{N} - \ddot{\theta}_L) - K (\frac{\dot{\theta}_M}{N} - \theta_L) = 0 \]
Laplace Transform:

\[
\begin{bmatrix}
J_M s^2 + (B_M + \frac{B}{N^2})s + \frac{K}{N^2} & -\frac{1}{N}(Bs + K) \\
-\frac{1}{N}(Bs + K) & \frac{J_L s^2 + (B + B_L)s + K}{D(s)}
\end{bmatrix}
\begin{bmatrix}
\Theta_M(s) \\
\Theta_L(s)
\end{bmatrix}
= \begin{bmatrix}
\hat{\Theta}_M(s) \\
\hat{\Theta}_L(s)
\end{bmatrix}
\]

Transfer functions:

\[
\frac{\Theta_M(s)}{T_M(s)} = \frac{\frac{J_L s^2 + (B + B_L)s + K}{D(s)}}{D(s)}
\]

\[
\frac{\Theta_L(s)}{T_M(s)} = \frac{-\frac{1}{N}(Bs + K)}{D(s)}
\]

\[
D(s) = (J_M J_L)s^4 + \left[(J_M + \frac{J_L}{N^2})B + J_M B_L + J_L B_M \right]s^3
\]

\[
+ \left[(J_M + \frac{J_L}{N^2})K + B_M B_L + B(B_M + \frac{B_L}{N^2}) \right]s^2 + \left[B_M + \frac{B_L}{N^2} \right]s
\]

\[
\frac{\Theta_L(s)}{\Theta_M(s)} = \frac{\frac{1}{N}(Bs + K)}{\frac{J_L s^2 + (B + B_L)s + K}{D(s)}}
\]
Let's look at a PI controller for $w_4$:

Let's assume: $B_n = B_L = 0$

$J_H = 5 \text{ kg-m}^2$

$J_L = 20 \text{ kg-m}^2$

$K = 7500 \frac{\text{N-m}}{\text{rad}}$

$\beta = 2 \frac{\text{N-m-s}}{\text{rad}}$

The bounds on the control gains will be functions of the plant parameters.

Goal here is to set bounds on the PI control gains, $K_p$ and $K_I$, to ensure robustness. A systematic procedure to tune the PI gains is sought. See "Robust PI Tuning for Elastic Two-Mass System" 1997 Calandri, Nordin, & Gutman.
Closed-Loop & Open-Loop Transfer Functions

\[ G_c(s) = K_p + \frac{K_e}{s} \]

\[ G(s) = H(s) \frac{K}{T(s)} = \frac{K}{s \left[ \frac{s^2}{\omega_n^2 + \frac{2\xi\omega_n}{\omega_n^2} + 1} \right]} \]

\[ K_1 = \frac{1}{J_H + J_L} = 0.04 \]

\[ \omega_n = \sqrt{\frac{K}{J_L}} = 19.4 \text{ rad/s} \]

\[ \omega_n = \sqrt{\frac{K(J_H + J_L)}{J_H J_L}} = 43.3 \text{ rad/s} \]

\[ \omega_n / \omega_n = 2.23 > 1 \quad \text{(always true.)} \]

Open-Loop TF: \[ G_c(s) G(s) \]

Closed-Loop TF: \[ \frac{G_c G}{1 + G_c G} \]
Open-Loop Transfer Function

$$\text{OL TF} = G_c(r) G(s)$$

$$G(s) = (0.04) \left[ \frac{s^2}{(19.4)^2} + \frac{2(0.0026)}{19.4} s + 1 \right]$$

$$\left( \frac{s}{(43.7)^2 + \frac{2(0.0058)}{43.7} s + 1} \right)$$

zeros: \(-0.05 \pm 19.4i\)
poles: \(0, -0.25 \pm 43.7i\)

PID controller: \(K_p + \frac{K_i}{s} = \frac{K_p s + K_i}{s} = \frac{K_p(s + \frac{K_i}{K_p})}{s}\)

zero: \(s = -\frac{K_i}{K_p}\)
pole: \(s = 0\)

A root-locus plot shows that this CL system is stable for any \(K_p, K_i\) gains.
How do we pick $K_p$ and $K_i$ control gains to avoid undamped oscillations due to the resonance/anti-resonance presence? These control gains will be bounded by functions of the plant parameters. From Galardini, Nordin, & Gutman, sufficient conditions for the rejection of undamped oscillations are:

$$K_i \leq \frac{K}{4} \frac{T_m + T_L}{T_L}$$

$$J_m K_p < K_p \leq 2 \sqrt{K_i (T_m + T_L)}$$

For the numerical values chosen, this reduces to:

$$K_i \leq 2343 \quad (K_i = 0 \text{ corresponds to proportional control})$$

$$216 + K_p \leq 484$$

Let's run simulations to see the results.
Simulation Study:

Let $K_p$ and $K_i$ be maximum values. \[
\begin{align*}
K_p &= 484 \\
K_i &= 2343
\end{align*}
\]
The open-loop system has:

- Pole: $0, -0.25 \pm 45.7i$
- Zero: $-0.05 \pm 19.4i$

PI Control \[ K_p \left( \frac{1}{s} + \frac{K_i}{K_p} \right) \]

Pole: $0$
Zero: $-\frac{K_i}{K_p}$
Gain: $K_p$

\[ \frac{K_i}{K_p} = 4.84 \]

Reference Input: Step at $t=0$, $0 \text{ to } 0.02 \text{ rad/s}$

Disturbance Torque: $T_d = 1 \text{ N-m at } t = 3 \text{ sec}$

Step input
As if we pick \( K_I = \frac{2545}{2} = 1172 \),

midrange \( K_p = 350 \)

Then PI controller:

- Pole: 0
- Zero: \(-\frac{K_I}{K_p} = -3.75\)
- Gain: \( K_p = 350 \)
Simulink Block Diagram
Motor Speed Control
PI Control

Motor - Load System with Flexible Coupling

Motor_Speed
To Workspace1

Load_Speed
To Workspace

Step1
Load Disturbance

K_P\cdot s + K_I
s
Transfer Fcn3

1
\frac{1}{J_M\cdot s + B_M}
Transfer Fcn

B\cdot s + K
s
Transfer Fcn1

1
\frac{1}{J_L\cdot s + B_L}
Transfer Fcn2

Load_Speed
To Workspace

Motor - Load System with Flexible Coupling
Root Locus Editor for Open Loop 1 (OL1)

$K_P = 484$
$K_I = 2343$

Root Locus Plot
Load Speed

Motor Speed

K_P = 484
K_I = 2343
$K_P = 350$
$K_I = 1172$
Load Speed

Motor Speed

K_P = 350
K_I = 1172