Modeling of Compliance

\[ J_M = \text{rotor inertia of a motor} \]
\[ J_L = \text{driven-load inertia} \]
\[ K_S = \text{elasticity of coupling} \]
\[ B_{ML} = \text{viscous damping of coupling} \]
\[ B_M = \text{viscous damping between ground and motor rotor} \]
\[ B_L = \text{viscous damping between ground and load inertia} \]
\[ T = \text{electromagnetic torque applied to motor rotor} \]
• Comments

– $K_S$, the elasticity of the coupling; it is often neglected in low-power systems; modeling it in high-power systems is essential.

– $B_{ML}$, the viscous damping of the coupling; it is usually small, as transmission materials provide little damping.

– $B_M$ and $B_L$ can be neglected in the following analysis, as they have a small effect on resonance. They are included here for completeness.

– Coulomb friction has been neglected. The fixed value of Coulomb friction has little impact on stability when the motor is moving. At rest, the impact of stiction on resonance is more complex. Sometimes stiction is thought of as increasing the load inertia when the motor is at rest. This accounts for the tendency of systems to change resonance behavior when the motion stops.
**Equations of Motion**

\[
\begin{align*}
T - B_M \ddot{\theta}_M - B_{ML} \left( \dot{\theta}_M - \dot{\theta}_L \right) - K_S \left( \theta_M - \theta_L \right) &= J_M \dddot{\theta}_M \\
-B_L \ddot{\theta}_L + B_{ML} \left( \dot{\theta}_M - \dot{\theta}_L \right) + K_S \left( \theta_M - \theta_L \right) &= J_L \dddot{\theta}_L
\end{align*}
\]

**Compliance Modeling**
Equations of Motion

\[ T - B_M \dot{\theta}_M - B_{ML} (\dot{\theta}_M - \dot{\theta}_L) - K_S (\theta_M - \theta_L) = J_M \ddot{\theta}_M \]
\[ -B_L \dot{\theta}_L + B_{ML} (\dot{\theta}_M - \dot{\theta}_L) + K_S (\theta_M - \theta_L) = J_L \ddot{\theta}_L \]

Laplace Transform of the Equations of Motion

\[
\begin{bmatrix}
J_M s^2 + (B_{ML} + B_M) s + K_S \\
-J_L s^2 + (B_{ML} + B_L) s + K_S
\end{bmatrix}
\begin{bmatrix}
\Theta_M (s) \\
\Theta_L (s)
\end{bmatrix}
= \begin{bmatrix}
(B_{ML} s + K_S) \\
(B_{ML} s + K_S)
\end{bmatrix}
\begin{bmatrix}
\dot{\Theta}_M (s) \\
\dot{\Theta}_L (s)
\end{bmatrix} + \begin{bmatrix}
T (s) \\
0
\end{bmatrix}
\]
MatLab / Simulink Block Diagram \((B_M = 0 \text{ and } B_L = 0)\)

\[
T - B_{ML} (\dot{\theta}_M - \dot{\theta}_L) - K_S (\theta_M - \theta_L) = J_M \ddot{\theta}_M
\]

\[
B_{ML} (\dot{\theta}_M - \dot{\theta}_L) + K_S (\theta_M - \theta_L) = J_L \ddot{\theta}_L
\]
Transfer Functions

\[
\frac{\Theta_M}{T}(s) = \frac{J_L s^2 + (B_{ML} + B_L)s + K_S}{D(s)}
\]

\[
\frac{\Theta_L}{T}(s) = \frac{B_{ML}s + K_S}{D(s)}
\]

\[
D(s) = [J_M J_L]s^4 + [(J_M + J_L)B_{ML} + J_M B_L + J_L B_M]s^3 + [(J_M + J_L)K_S + B_M B_L + B_{ML}(B_L + B_M)]s^2 + [(B_M + B_L)K_S]s
\]
Transfer Functions
(B_L = 0 and B_M = 0)

\[
\Theta_M(s) = \frac{1}{(J_M + J_L)s^2} \left[ \frac{J_L s^2 + B_{ML} s + K_S}{J_L + J_M} \right]
\]

\[
\Theta_L(s) = \frac{1}{(J_M + J_L)s^2} \left[ \frac{B_{ML} s + K_S}{J_L J_M s^2 + B_{ML} s + K_S} \right]
\]

As \( K_S \to \infty \) or as \( s \to 0 \) \( \Theta_M(s) = \Theta_L(s) = \frac{1}{(J_M + J_L)s^2} \) Rigid-Body Motion
Transfer Functions in Standard Form
(B_M = 0 and B_L = 0)

\[
\frac{\Theta_M}{T}(s) = \frac{K}{s^2} \left[ \frac{s^2}{\omega^2_{AR}} + 2\zeta_{AR} \frac{s}{\omega_{AR}} + 1 \right]
\]

\[
\frac{\Theta_L}{T}(s) = \frac{K(\tau s + 1)}{s^2} \left[ \frac{s^2}{\omega^2_{AR}} + 2\zeta_{AR} \frac{s}{\omega_{AR}} + 1 \right]
\]

Natural frequency of load connected to ground through the compliance

\[
K = \frac{1}{J_M + J_L}
\]

\[
\tau = \frac{B_{ML}}{K_S}
\]

\[
\omega_R = \sqrt{\frac{K_S (J_M + J_L)}{J_M J_L}}
\]

\[
\zeta_R = \frac{B_{ML}}{2 \sqrt{K_S J_M J_L}}
\]

\[
\omega_{AR} = \sqrt{\frac{K_S}{J_L}}
\]

\[
\zeta_{AR} = \frac{B_{ML}}{2 \sqrt{K_S J_L}}
\]
Compliance Modeling

Sample Values:
- \( J_L = 0.002 \text{ kg-m}^2 \)
- \( J_M = 0.002 \text{ kg-m}^2 \)
- \( K_S = 200 \text{ N-m/rad} \)
- \( B_{ML} = 0.01 \text{ N-m-s/rad} \)

\( \omega_{AR} = 316 \text{ rad/s} = 50.3 \text{ Hz} \)
\( \omega_R = 447 \text{ rad/s} = 71.2 \text{ Hz} \)
\( \zeta_{AR} = 0.008 \)
\( \zeta_R = 0.011 \)

\( \frac{\Theta^M}{T} (s) \)

\( \Omega M (s) \)

Compliantly-Coupled Motor + Load

\( \omega_R > \omega_{AR} \) always

\[ \frac{1}{(J_M + J_L)s^2} \]

Rigidly-Coupled Motor + Load

\[ \frac{1}{J_M s^2} \]
\[ \frac{\Theta_M}{T}(s) = \left[ \frac{1}{(J_M + J_L)s^2} \right] \left[ \frac{J_Ls^2 + B_MLs + K_S}{J_LJ_Ms^2 + B_MLs + K_S} \right] \]

Collocated System \( \omega_R > \omega_{AR} \) always

Bode Diagram

Rigidly-Coupled Motor + Load
\[ \frac{\Theta_L}{T}(s) = \left[ \frac{1}{(J_M + J_L)s^2} \right] \frac{B_{ML}s + K_S}{\frac{J_LJ_M}{J_L + J_M}s^2 + B_{ML}s + K_S} \]

Non-Collocated System

Compliance Modeling
\[ \frac{\Theta_M(s)}{T(s)} \quad \frac{\Theta_L(s)}{T(s)} \]

Bode Plot Comparison

Compliance Modeling
\[
\frac{\Theta_L}{\Theta_M}(s) = \frac{(\tau s + 1)}{\left[ \frac{s^2}{\omega^2_{AR}} + \frac{2\zeta_{AR} s}{\omega_{AR}} + 1 \right]}
\]

Note: There is no anti-resonant frequency in this transfer function. Also, there is 90\(^\circ\) more phase lag at high frequency.
Effect of $J_L / J_M$ Ratio on Resonance and Anti-Resonance

\[
\frac{J_L}{J_M} = 1
\]

\[
\frac{J_L}{J_M} = 5
\]

\[
\frac{J_L}{J_M} = \frac{1}{5} \frac{\Theta_M}{T} (s)
\]

$J_M = 0.002 \text{ kg-m}^2$

$K_S = 200 \text{ N-m/rad}$

$B_{ML} = 0.01 \text{ N-m-s/rad}$

$\omega_R \text{ lower limit} = 316 \text{ rad/sec}$

$\omega_{AR} \text{ lower limit} = 0$

Compliance Modeling
Effect of Varying $\frac{J_L}{J_M}$

\[
\Theta_M(s) \frac{1}{T(s)}
\]
\[ \frac{J_L}{J_M} = 100, 10, 1, 0.1 \]

\[ \Theta_M(s) \]

\[ T(s) \]

\[ \omega_R \text{ lower limit} = 316 \text{ rad/sec} \]

Compliance Modeling

K. Craig
\[ \frac{J_L}{J_M} = 100, 1000 \]

\[ \frac{\Theta_M(s)}{T(s)} \]

Bode Diagram

- \( \omega_R \) lower limit = 316 rad/sec

Compliance Modeling
Effect of $K_S$ on Resonance and Anti-Resonance

\[
\frac{J_L}{J_M} = 1
\]

\[
K_S = 20 \ \text{N-m rad}
\]

\[
K_S = 200 \ \text{N-m rad}
\]

\[
K_S = 2000 \ \text{N-m rad}
\]

\[
\frac{\Theta_M}{T}(s)
\]
Limiting Behavior:

\[
\frac{\Theta_M}{T}(s) = \left[ \frac{1}{(J_M + J_L)s^2} \right] \left[ \frac{J_L s^2 + B_{ML}s + K_S}{J_L J_M s^2 + B_{ML}s + K_S} \right]
\]

\approx \frac{1}{(J_M + J_L)s^2} \quad \text{as } \omega \to 0

\approx \frac{1}{J_M s^2} \quad \text{as } \omega \to \infty

\omega_R = \sqrt{\frac{K_S (J_M + J_L)}{J_M J_L}} \quad \text{As } J_L \to \infty, \quad \omega_{AR} \to 0 \quad \text{and} \quad \omega_R \to \sqrt{\frac{K_S}{J_M}}

\omega_{AR} = \sqrt{\frac{K_S}{J_L}}
• **Observations**
  
  – For \( J_M > 0 \), the anti-resonance frequency always occurs before the resonance frequency.

  \[
  \omega_R = \sqrt{\frac{K_S (J_M + J_L)}{J_M J_L}}
  \]

  \[
  \omega_{AR} = \sqrt{\frac{K_S}{J_L}}
  \]

  – At a low \( J_L/J_M \) ratio, the resonance and anti-resonance frequencies are close to each other at a high frequency.
- As \( J_L / J_M \) increases, both the anti-resonance and resonance frequency decrease, with the anti-resonance frequency decreasing at a faster rate.

\[
\omega_R = \sqrt{\frac{K_S (J_M + J_L)}{J_M J_L}} = \sqrt{\frac{1 + \frac{J_L}{J_M}}{K_S J_L}} = \sqrt{\frac{1 + \frac{J_M}{J_L}}{K_S J_M}}
\]

\[
\omega_{AR} = \sqrt{\frac{K_S}{J_L}} = \sqrt{K_S \frac{1}{J_L}}
\]

- At \( J_L = J_M \), \( \omega_{AR} = 0.707 \omega_R \).

As \( J_L \to \infty \), \( \omega_{AR} \to 0 \) and \( \omega_R \to \sqrt{\frac{K_S}{J_M}} \).
– For a given $J_L$, to increase the resonance frequency, either increase the shaft stiffness or decrease the motor inertia.

\[
\omega_R = \sqrt{\frac{K_S (J_M + J_L)}{J_M J_L}} = \sqrt{K_S \frac{1 + \frac{J_L}{J_M}}{J_L}} = \sqrt{K_S \frac{1 + \frac{J_M}{J_L}}{J_M}}
\]

– As $K_S$ increases, both $\omega_R$ and $\omega_{AR}$ increase.

– A general guideline to avoid instability problems is to keep the desired closed-loop bandwidth well below the resonance frequency and the ratio $J_L / J_M$ less than 5.
- **Heavy-Load Approximation**: $J_L > 5J_M$

\[
\frac{J_M + J_L}{J_M + J_L} \approx J_L
\]

\[
\frac{J_M J_L}{J_M + J_L} = \frac{J_M}{J_M + J_L} \approx \frac{J_M}{J_L} + 1
\]

\[
\frac{\Theta_M}{T}(s) = \left[ \frac{1}{(J_M + J_L)s^2} \right] \left[ \frac{J_L s^2 + B_{ML} s + K_S}{J_L J_M s^2 + B_{ML} s + K_S} \right]
\]

\[
\approx \frac{1}{J_L s^2} \left[ \frac{J_L s^2 + B_{ML} s + K_S}{J_M s^2 + B_{ML} s + K_S} \right]
\]
Heavy-Load Approximation

\[ \frac{J_L}{J_M} = 5 \]
Belt-Driven Load and Motor

Rigid Belt Case:

\[ R_M \theta_M = R_L \theta_L \]

\[ \left[ J_M + \left( \frac{R_M}{R_L} \right)^2 J_L \right] \ddot{\theta}_M = T \]
\[ J_M \ddot{\theta}_M + B_M \dot{\theta}_M + 2K_R^2 \theta_M + 2B_R^2 \dot{\theta}_M = T + 2K_R R_L \theta_L + 2B_R R_L \dot{\theta}_L \]

\[ J_L \ddot{\theta}_L + B_L \dot{\theta}_L + 2K_R^2 \theta_L + 2B_R^2 \dot{\theta}_L = 2K_R R_L \theta_M + 2B_R R_L \dot{\theta}_M - T_L \]
Equations of Motion

\[ J_M \ddot{\theta}_M + B_M \dot{\theta}_M + 2K_R^2 \theta_M + 2B_R^2 \dot{\theta}_M = T + 2K_R R L \theta_L + 2B_R R L \dot{\theta}_L \]

\[ J_L \ddot{\theta}_L + B_L \dot{\theta}_L + 2K_R^2 \theta_L + 2B_R^2 \dot{\theta}_L = 2K_R R L \theta_M + 2B_R R L \dot{\theta}_M - T_L \]

Laplace Transform of the Equations of Motion \((T_L = 0)\)

\[
\begin{bmatrix}
J_M s^2 + (B_M + 2B_R^2) s + 2K_R^2 \\
J_L s^2 + (B_L + 2B_R^2) s + 2K_R^2
\end{bmatrix}
\begin{bmatrix}
\Theta_M(s) \\
\Theta_L(s)
\end{bmatrix} = 
\begin{bmatrix}
(2B_R R_L s + 2K_R R_L) \Theta_L(s) + T(s) \\
(2B_R R_L s + 2K_R R_L) \Theta_M(s)
\end{bmatrix}
\]

\[
\begin{bmatrix}
J_M s^2 + (B_M + 2B_R^2) s + 2K_R^2 \\
- (2B_R R_L s + 2K_R R_L)
\end{bmatrix}
\begin{bmatrix}
\Theta_M(s) \\
\Theta_L(s)
\end{bmatrix} = 
\begin{bmatrix}
T(s) \\
0
\end{bmatrix}
\]
Transfer Functions

\[ \frac{\Theta_M}{T}(s) = \frac{J_L s^2 + (B_L + 2BR_L^2)s + 2KR_L^2}{D(s)} \]

\[ \frac{\Theta_L}{T}(s) = \frac{2BR_M R_L s + 2KR_M R_L}{D(s)} \]

\[ D(s) = [J_M J_L]s^4 + \left[ 2B(J_L R_M^2 + J_M R_L^2) + J_M B_L + J_L B_M \right]s^3 + \left[ 2K(J_L R_M^2 + J_M RL) + B_M B_L + 2B(B_L R_M^2 + B_M R_L^2) \right]s^2 + \left[ 2K(B_L R_M^2 + B_M R_L^2) \right]s \]
Transfer Functions
\((B_L = 0 \text{ and } B_M = 0)\)

\[
\Theta_M(s) = \frac{J_L s^2 + 2BR_L^2 s + 2KR_L^2}{[J_M J_L] s^4 + [2B(J_L R_M^2 + J_M R_L^2)] s^3 + [2K(J_L R_M^2 + J_M R_L^2)] s^2}
\]

\[
\Theta_L(s) = \frac{2BR_M R_L s + 2KR_M R_L}{[J_M J_L] s^4 + [2B(J_L R_M^2 + J_M R_L^2)] s^3 + [2K(J_L R_M^2 + J_M R_L^2)] s^2}
\]
Transfer Functions
(BL = 0 and BM = 0)

\[
\Theta_M(s) = \frac{1}{(J_L R_M^2 + J_M R_L^2)s^2} \left[ \frac{J_L s^2 + 2BR_L^2 s + 2KR_L^2}{\frac{J_M J_L}{J_L R_M^2 + J_M R_L^2} s^2 + 2Bs + 2K} \right]
\]

\[
\Theta_L(s) = \frac{1}{(J_L R_M^2 + J_M R_L^2)s^2} \left[ \frac{2BR_M R_L s + 2KR_M R_L}{\frac{J_M J_L}{J_L R_M^2 + J_M R_L^2} s^2 + 2Bs + 2K} \right]
\]
Transfer Functions in Standard Form
(B\textsubscript{M} = 0 and B\textsubscript{L} = 0)

\[ \Theta\textsubscript{M}(s) = \frac{K_1}{s^2 \left[ \frac{s^2}{\omega_{AR}^2} + \frac{2\zeta_{AR}s}{\omega_{AR}} + 1 \right]} \]

\[ \Theta\textsubscript{L}(s) = \frac{K_2 (\tau s + 1)}{s^2 \left[ \frac{s^2}{\omega_R^2} + \frac{2\zeta_Rs}{\omega_R} + 1 \right]} \]

\[ K_2 = \frac{R_M R_L}{J_M R_M^2 + J_L R_L^2} \]

\[ K_1 = \frac{R_L^2}{J_M R_L^2 + J_L R_M^2} \quad \tau = \frac{B}{K} \]

\[ \omega_R = \sqrt{\frac{2K (J_M R_L^2 + J_L R_M^2)}{J_M J_L}} \]

\[ \zeta_R = \frac{B}{\sqrt{2K J_M J_L}} \]

\[ \omega_{AR} = \sqrt{\frac{2KR_L^2}{J_L}} \]

\[ \zeta_{AR} = \frac{BR_L}{\sqrt{2K J_L}} \]